DEPARTMENT OF PHYSICS
FU JEN UINVERSITY

# TOPICS ON PRECALCULUS

工欲善其事，
必先利其器！

August 2009

## 一個重要的訊息

歡迎各位來到輔大物理系！這個小冊子希望能幫助你在開始修[普通物理]之前做好準備。這門課非常重要，是所有高年級進階課程的基礎。對你而言，在大一的時候有一個好的開始是很關鍵的；如果你這門課學得不好，你會很難應付大學裡其他的物理課程。

這個小冊子包含所有你在開始學[普通物理]之前所應具備的數學能力。請務必下功夫複習(或學習，如果你沒學過)這些內容，而且完成每一個章節的練習題。對於以書面方式完成習題的同學，將會獲得平時成績的加分。如果你有任何問題，請與我們聯絡。我們將在學期初舉行測驗，範圍是第一至六章。最後，再次提醒你，數學能力的準備對於是否能成功地學習[普通物理]這門課是非常重要的。

祝你好運！

## An important message

Welcome to the Physics Department of the Fu-Jen Catholic University! This brochure is intended to help you better prepare for the course of General Physics, which is the very foundation for all the advanced courses you will be taking in the next few years. We simply cannot stress enough how important it is for you to get off with a good start and do well in this course in your freshman year. If you do not do well in this course, it would be very difficult to go through the rest of your college study in physics.

This brochure contains the basic mathematical techniques that you **must** familiarize yourself with before beginning with the course. Please **DO** make the efforts to review and/or learn the contents, and complete the exercises in each chapter. Prizes will be given out to those who present his/her completed exercises in written. Feel free to contact us if you have any question. A quiz on the materials from Chapters 1-6 will be given at the beginning of the semester. Please note that being prepared with these mathematical techniques is essential for your success in the course of General Physics.

Good Luck!

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# Contents

# Chapter 1

# Coordinate system

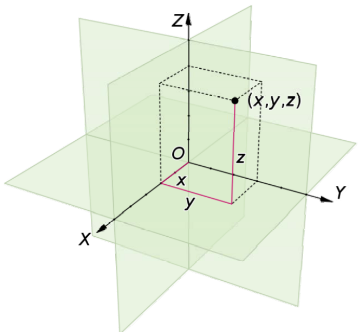
## 1.1 Coordinate and coordinate system

A coordinate is a number that determines the location of a point along some line or curve. A list of two or three coordinates can be used to determine the location of a point on a surface or in a volume. More coordinates are required to specify the location of a point in higher-dimensional domain.

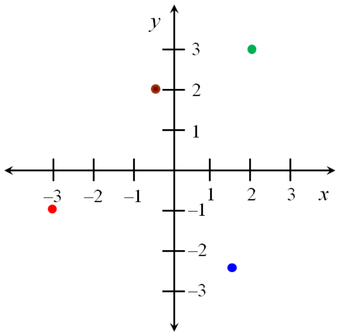
A two- and three-dimensional coordinate system is a system for assigning a 2- and 3-tuple of numbers or scalars to each point in a two- and three-dimensional space, respectively. An $n$-tuple of numbers is used to denote a point in an $n$-dimensional space.

## 1.2 Cartesian coordinate system

Cartesian coordinate system (also called rectangular coordinate system) is a rectilinear two- or three-dimensional coordinate system. The three axes of three-dimensional Cartesian coordinates, conventionally denoted as the $x$-, $y$-, and $z$-axes are chosen to be linear and mutually perpendicular and intersect at the origin. In three dimensions, the coordinates $x$, $y$, and $z$ may lie anywhere in the interval $(-\infty, \infty)$.



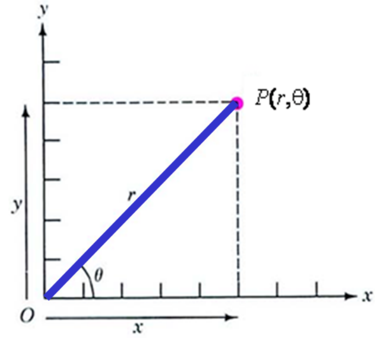
**Ex:** Find the coordinates of the following four points.




**Ex:** (a) Find the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. (b) Determine the equation of a straight line passing through two distinct points $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$. What is the slope of the line?

## 1.3 Polar coordinate system

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point (e.g., the origin) and an angle from a fixed direction (e.g., the +x-axis). The angle $\theta$, often called the polar angle, is measured counterclockwise from the fixed axis.

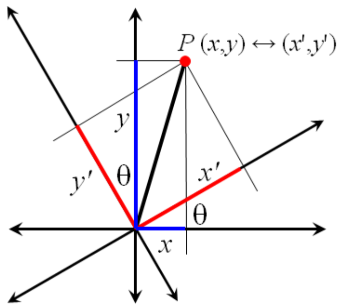



The polar coordinates $r$ and $\theta$ are defined in terms of Cartesian coordinates by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

## 1.4 Coordinate transformation

A coordinate transformation is a conversion from one coordinate system to another, to describe the same space. For example, the coordinate transformation for the coordinate system, $(x', y')$, which rotates the $x$- and $y$-axes with an angle $\theta$ in a counterclockwise direction when looking towards the origin is given by




$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta. \end{cases}$$

In terms of matrix, it can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Ex.** Find the coordinate transformation for $x$ and $y$ in terms of $x'$ and $y'$ for the previous example.
**Ex.** Find the coordinate transformation between the polar coordinates and the two-dimensional Cartesian coordinates. (See Sec.1.3.)

## 1.5 Physical significance

Coordinate transformation is very useful in physics, especially for two distinct observers to describe an identical event, when they are communicating with each other. With respect to different frames of reference, the event may have different temporal and spatial coordinates. Since the event is identical, the coordinates acquired by the two observers must be related to one another. It is the coordinate transformation that gives explicitly the mathematical relationship between these two seemingly distinguishable coordinates. For example, the final match of World Cup 2010, Netherlands vs Spain, kicked off at the Soccer City Stadium in Johannesburg, South Africa, at $20^{\underline{00}}$ on July 11, 2010. Here in Taiwan, it was about $11.5 \times 10^3$ km at an angle 28.7° south of west (SOW) from Taipei. Accordingly, we may denote the event taken place by coordinates $\mathbf{R}_t = (-10, -5.5) \times 10^3$ km with respect to Taipei at $T_t = 02^{\underline{00}}$ on July 12, 2010. On the other hand, Mr. C. M. Wang, who was in Washington DC, would describe it as $\mathbf{R}_w = (11, -6.9) \times 10^3$ km with respect to him at $T_w = 14^{\underline{00}}$ on July 11, 2010. The relation between the two coordinates can be written by $\mathbf{R}_t = \mathbf{R}_w + (-21, 1.4) \times 10^3$ km and $T_t = T_w + 12^{\underline{00}}$. It is worth to note that this example delivers primarily the idea of coordinate transformation except that the spherical nature of the globe has not been correctly taken into consideration.

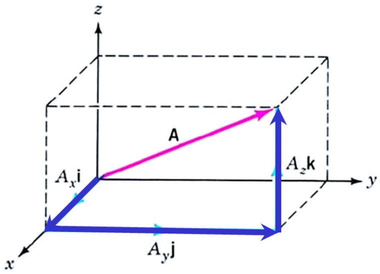
# Chapter 2

# Vector

## 2.1 Definition

A vector is a geometric object that is specified by both a magnitude and a direction in space. A 3-dimensional vector is represented by 3 coordinates.

1. In the Cartesian coordinate system, it can be expressed componentwise as $\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ or in terms of unit vectors as $\mathbf{A} = A_1\,\hat{\mathrm{i}} + A_2\,\hat{\mathrm{j}} + A_3\,\hat{\mathrm{k}}$.[1]

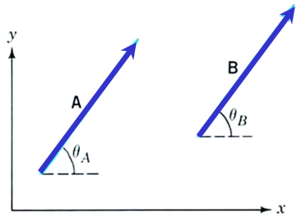



2. When written out componentwise, the notation $\mathbf{r}$ generally refers to $\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$. On the other hand, when written with a subscript, the notation $\mathbf{r}_1$ generally refers to $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$.

3. The length (magnitude) of a vector is denoted as $A = |\mathbf{A}|$ and defined by $|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$.

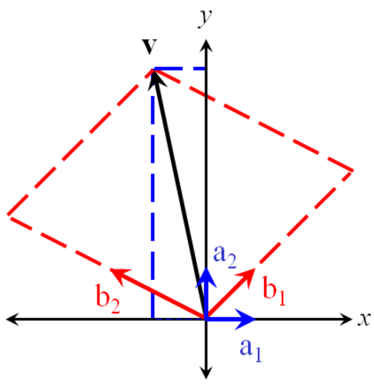
[1]For unit vectors, see Sec. 2.3.

4. Vector equality: two vectors are equal, $\mathbf{A} = \mathbf{B}$, if and only if $A = B$ and $\theta_A = \theta_B$.




## 2.2 Bases and representations[2]

A basis for an $n$-dimensional vector space is a set of $n$ linearly independent vectors that every vector in the given vector space can can be expressed uniquely as a linear combination of the basis vectors, and that no element of the set can be represented as a linear combination of the others. The choice of a basis for an $n$-dimensional vector space is, however, unlimited. Nevertheless, there are always the same number of basis vectors in each of them. Moreover, two linearly independent bases are related to each other by a linear transformation.




For example, two sets of vectors $A = \{\mathbf{a}_1, \mathbf{a}_2\}$ with $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ can be used as bases for the 2-dimensional vector space. Then, the vector $\mathbf{v}$ is expressed in terms of bases $A$ and $B$ as $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}_A$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}_B$, respectively. The former, $\begin{pmatrix} -1 \\ 5 \end{pmatrix}_A$, is the representation of the vector $\mathbf{v}$ in basis $A$ and the latter, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}_B$, the representation of $\mathbf{v}$ in basis $B$.

As mentioned previously, the change of basis from $B$ to $A$ can be performed by a

[2]Formal treatment of basis transformation will be given in the course of "*Matrices and Vectors*".

linear transformation. It is easy to show that

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}.$$

Consequently, the representations of the vector $\mathbf{v}$ in the two bases are related by

$$\begin{pmatrix} -1 \\ 5 \end{pmatrix}_A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}_B.$$

**Ex.** (a) Find the linear transformation for the change of basis from $A$ to $B$. (b) Find the matrix $M$ such that $\begin{pmatrix} 3 \\ 2 \end{pmatrix}_B = M \begin{pmatrix} -1 \\ 5 \end{pmatrix}_A$.

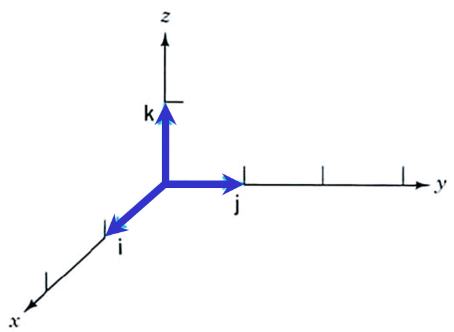
**Ex.** For an arbitrary vector $\mathbf{u}$, can you find the relation between the two representations $\begin{pmatrix} u_x \\ u_y \end{pmatrix}_B = M \begin{pmatrix} u'_x \\ u'_y \end{pmatrix}_A$, ie., find the matrix $M$?

**Ex.** Find the representation of $\mathbf{v}$ in terms of the basis $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ with $\mathbf{c}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

**Ex.** Can the two vectors $\mathbf{d}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a basis? Explain.

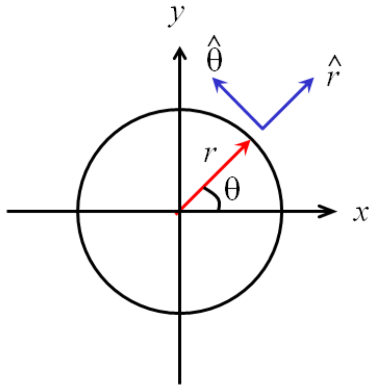
## 2.3 Unit vector

A unit vector is a vector of length 1 (the unit length) and defined by $\hat{\mathrm{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$. In the three-dimensional Cartesian coordinate system, the unit vectors co-directional with the $x$, $y$, and $z$ axes are given by $\hat{\mathrm{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\mathrm{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\hat{\mathrm{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.




**Ex.** Find the unit vector of $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

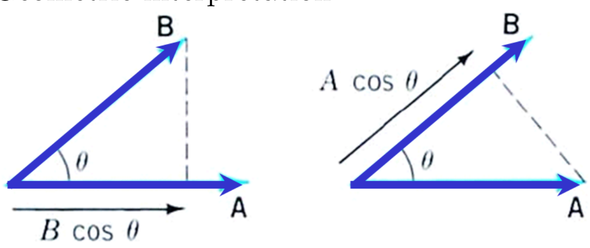
**Ex.** Find the unit vectors $\hat{r}$ and $\hat{\theta}$ for the polar coordinate system.




## 2.4 Scalar (dot) product

The scalar product, also known as the dot product or the inner product, is an operation of mapping two vectors onto a real-valued scalar quantity.

1. The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = \mathbf{B}\cdot\mathbf{A}$, where $\theta$ is the angle between the vectors.
2. The length of a vector can then be determined by $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$.
3. Accordingly, the dot product of two vectors can be given by
$$\mathbf{A}\cdot\mathbf{B} = \mathbf{A}^T\mathbf{B} = (A_1, A_2, A_3)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = A_1B_1 + A_2B_2 + A_3B_3.$$
4. Geometric interpretation



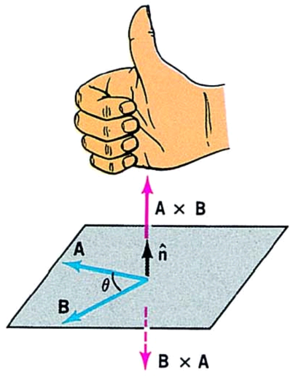

$B\cos\theta$ is the scalar projection of $\mathbf{B}$ onto $\mathbf{A}$, whereas $A\cos\theta$ is the scalar projection of $\mathbf{A}$ onto $\mathbf{B}$.

5. It follows immediately that $A_1 = \hat{\mathrm{i}}\cdot\mathbf{A}$, $A_2 = \hat{\mathrm{j}}\cdot\mathbf{A}$, and $A_3 = \hat{\mathrm{k}}\cdot\mathbf{A}$.
6. Orthogonality: two non-zero vectors $\mathbf{A}$ and $\mathbf{B}$ are said to be orthogonal (perpendicular to each other) if and only if $\mathbf{A}\cdot\mathbf{B} = 0$.

**Ex.** Show that $\hat{\mathrm{i}}\cdot\hat{\mathrm{i}} = 1 = \hat{\mathrm{j}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{k}}$ and $\hat{\mathrm{i}}\cdot\hat{\mathrm{j}} = 0 = \hat{\mathrm{i}}\cdot\hat{\mathrm{k}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{k}}$.
**Ex.** Are the unit vectors of polar coordinate system orthogonal?

## 2.5 Vector (cross) product

The vector product, also known as the cross product, is a binary operation of mapping two vectors in a three-dimensional Euclidean space that results in another vector which is perpendicular to the plane containing the two input vectors.

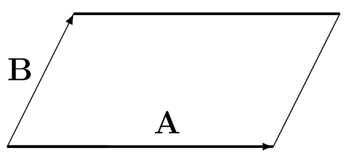



1. The cross product of two vectors of $\mathbf{A}$ and $\mathbf{B}$ is defined as $\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathrm{n}} = -\mathbf{B}\times\mathbf{A}$, where $\theta$ is the angle between the vectors measured from $\mathbf{A}$ to $\mathbf{B}$. $\hat{\mathrm{n}}$ is a unit vector normal to the plane containing $\mathbf{A}$ and $\mathbf{B}$ given by the right hand rule as illustrated in the figure.
   **Ex.** Show that (a) $\hat{\mathrm{i}}\times\hat{\mathrm{i}} = 0 = \hat{\mathrm{j}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}\times\hat{\mathrm{k}}$; (b) $\hat{\mathrm{i}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}$, $\hat{\mathrm{j}}\times\hat{\mathrm{k}} = \hat{\mathrm{i}}$, and $\hat{\mathrm{k}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}$.

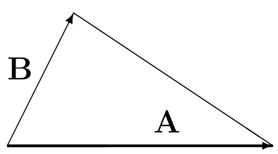
2. It is straightforward to calculate the cross product of two 3-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$ in terms of their components and found that for $\mathbf{C} = \mathbf{A}\times\mathbf{B}$, $C_x = A_yB_z - A_zB_y$, $C_y = A_zB_x - A_xB_z$, and $C_z = A_xB_y - A_yB_x$. These can be written in a shorthand notation that takes the form of a determinant
$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathrm{i}}(A_yB_z - A_zB_y) + \hat{\mathrm{j}}(A_zB_x - A_xB_z) + \hat{\mathrm{k}}(A_xB_y - A_yB_x).$$

3. Geometric interpretation:

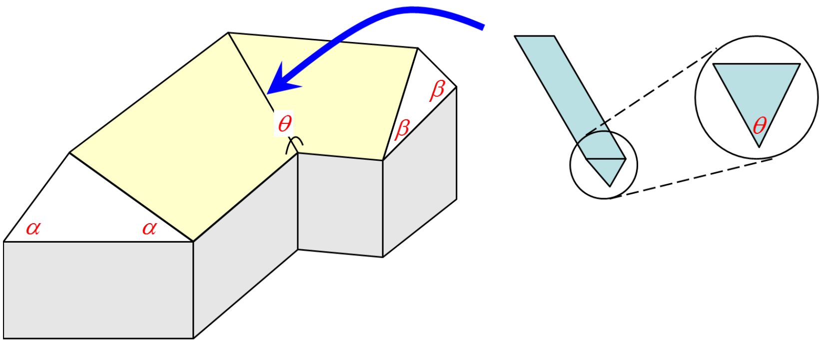
   (a) The area of a parallelogram with adjacent side lengths of $A$ and $B$ can be determined by $|\mathbf{A}\times\mathbf{B}|$.

   (b) The area of a triangle with two adjacent side lengths of $A$ and $B$ can be obtained from $\frac{1}{2}|\mathbf{A}\times\mathbf{B}|$.

**Ex.** A farmhouse consists of two wings connected at right angle as shown in the figure. The side view of left wing can be regarded as an isosceles triangle with isosceles angle $\alpha = 60°$ atop a rectangle, whereas that of the right wing as an isosceles triangle with isosceles angle $\beta = 45°$ upon a rectangle. For a house renovation, it requires a replacement of the valley flashing. To do this, it is necessary to make a wedge-shaped timber. Estimate the top angle of the timber.




## 2.6 Physical significance

Many physical quantities exhibit characteristics of a vector. For instance, the quantity **displacement** that we shall encounter very soon in the class, representing the change in position, is a vector. It depends only on the initial and final positions, not the path taken. Vector components vary with the change of coordinate axes, however, the magnitude as well as the direction of the vector remains unchanged. Therefore, vector equation retains its form regardless of coordinate system. This reflects the fact that **physical laws are coordinate system independent**.

# Chapter 3

# Functions

## 3.1 Definition

A function is a mapping of members of one set into members of another set. The notation $f : A \mapsto B$ from $A$ to $B$ is a function $f$ such that for every $a \in A$, there is an object $f(a) \in B$. A slightly different notation $f : x \to f(x)$ specifies that $f$ is a function acting upon a number $x$ and returning a value $f(x)$.

Consider a function $f(x) = \dfrac{3}{x^2+1}$. Complete the following table using your calculator.

| $x$ | -10 | -5 | -2.5 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.5 | 5 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | | | | | | | |

**Ex.** (a) Use these data to plot $f(x)$ versus $x$. (b) What is the maximum value of $f(x)$? (c) What is the value of $x$ corresponding to the maximum? (d) As $|x|$ becoming large, how can you depict the behavior of $f(x)$?

Quite often, if not always, plotting a function in dependence of the variable is quite helpful. It enables us to get an idea about the behavior of the function.

## 3.2 Polynomial

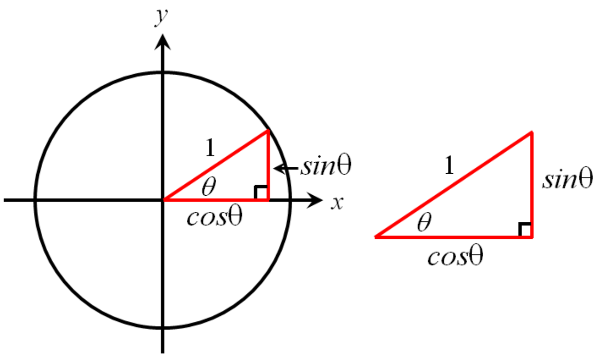
A polynomial is a mathematical expression involving a sum of non-negative whole number exponents in one or more variables multiplied by coefficients. A one variable polynomial is given by $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$. The highest power in a single variable polynomial is called its order, or sometimes its degree. The polynomial of degree $n$ can be abbreviated with the summation $\sum_{r=0}^{n} a_r x^r$.

**Ex.** Which of the followings are not polynomials? (a) $x^2 + x - x^{1/2} + 2$; (b) $2x^2 - \frac{3}{x} + 1$; (c) $3x^2 - 2x + 3$; (d) $2^x - x + 1$.

## 3.3 Trigonometric functions

The six trigonometric functions of sine ($\sin x$), cosine ($\cos x$), tangent ($\tan x$), cotangent ($\cot x$), cosecant ($\csc x$), and secant ($\sec x$) are well known and among the most frequently used elementary functions. In fact, only $\sin x$ and $\cos x$ need to be defined, the other four can be derived from the sine and cosine functions.

1. For $\theta$ being an angle measured counterclockwise from the $x$-axis along an arc of the unit circle, then $\sin\theta$ and $\cos\theta$ are, respectively, the vertical and the horizontal coordinates of the arc endpoint, as illustrated in the left figure below. These extend the schoolbook definitions of the sine and cosine of an angle $\theta \in [0, \pi/2]$ to $\theta \in \mathbb{R}$.




2. The other four trigonometric functions are defined as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$

3. Useful identities:

   (a) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

   (b) $\sin^2\theta + \cos^2\theta = 1$

   (c) $1 + \tan^2\theta = \sec^2\theta$

   (d) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

   (e) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

## 3.4 Exponential function

The exponential function defined as $\exp(x) = e^x$, where $e = 2.71828182846\ldots$, is the unique solution to the differential equation $\dfrac{df(x)}{dx} = f(x)$ with the initial condition $f(0) = 1$.[1]

Below list some useful properties of $e^x$:

1. $e^{x+y} = e^x e^y$

2. In virtue of the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ with $i = \sqrt{-1}$ the exponential function relates to the trigonometric functions as the following

   (i) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and

   (ii) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

   These extend further the trigonometric functions for complex arguments, i.e., $\theta \in \mathbb{C}$. As a result, one can easily show

   (i) $\cos(iy) = \dfrac{e^{y} + e^{-y}}{2} = \cosh y$

   (ii) $\sin(iy) = -\dfrac{e^{y} - e^{-y}}{2i} = i\sinh y$

3. The limit definition of the exponential function is given by

   $e^x = \lim_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n$ for any $x \in \mathbb{R}$.

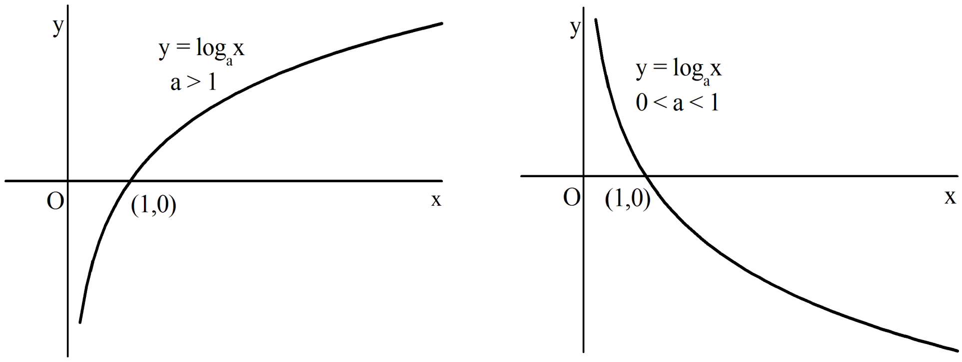
   Details on this approach will be discuss in the Calculus course.

## 3.5 Logarithm function

The logarithm function $y = f(x) = \log_a x$ for a base $a > 0$ $(a \neq 1)$ and a number $x$ is defined to be the inverse function of the exponential function, i.e., $x = f^{-1}(y) = a^y$. More explicitly, the logarithm of a number $(x)$ to a given base $(a)$ is the power or exponent $(y)$ to which the base must be raised in order to produce the number. For example, the logarithm of 100 to base 10 is 2, since $10^2 = 100$, and the logarithm of 32 to base 2 is 5 because $2^5 = 32$. Consequently, for any $x$ and $a > 0$ it give rise to $x = \log_a(a^x)$, or equivalently, $x = a^{\log_a x}$.

The function $\log_a x$ depends on both $a$ and $x$, but the term logarithm function in standard usage refers to a function of the form $\log_a x$ in which the base $a$ is fixed and so the only argument is $x$. Apparently, $\log_a x$ is a monotonic function of $x$. It is increasing if $a > 1$ and decreasing if $a < 1$ as shown in the following figures.

[1]For the derivative of a function, see Sec. 5.2.1.




If the base of the logarithm function is chosen as 10, it is called the **common logarithm** and denoted by $\log x = \log_{10} x$. On the other hand, the logarithm function to base "$e$" is called the **natural logarithm** and denoted by $\ln x = \log_e x$.

Below are some useful relations.

1. Logarithmic identities
   ($a$) $\log_a(xy) = \log_a x + \log_a y$ cf. $(a^x)(a^y) = a^{x+y}$
   ($b$) $\log_a(x/y) = \log_a x - \log_a y$ cf. $a^x/a^y = a^{x-y}$
   ($c$) $\log_a x^y = y \log_a x$ cf. $(a^x)^y = a^{xy}$
   ($d$) $\log_a \sqrt[y]{x} = \frac{1}{y} \log_a x$ cf. $\sqrt[y]{a^x} = a^{x/y}$

2. Change of base
   $$\log_a x = \frac{\log_b x}{\log_b a}$$

# Chapter 4

# Limit

## 4.1 Examples

**(1a)** Consider $f(x) = \dfrac{x}{x^2+1}$ as $x$ approaches 2. Complete the following table using your calculator.

| $f(1.9)$ | $f(1.99)$ | $f(1.999)$ | $f(1.9999)$ | $f(2)$ | $f(2.0001)$ | $f(2.001)$ | $f(2.01)$ | $f(2.1)$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

As you might guess from the example, as $x$ approaches the value of 2 from above or below, the value of the function $f(x)$ approaches the value of $f(2) = 0.4$. In the case, we can say that $f(x)$ have a **limit** of 0.4 as $x$ is made very close to 2.

**(1b)** Find the value of $\epsilon$, such that $|f(x) - 0.4| < 0.01$ for $x \in [2-\epsilon, 2+\epsilon]$.

**(1c)** Find the value of $\epsilon$, such that $|f(x) - 0.4| < 0.0001$ for $x \in [2-\epsilon, 2+\epsilon]$.

**(2a)** Consider $g(x) = \dfrac{x^2-4}{\sqrt{x^2}-2}$ as $x$ approaches 2. Complete the following table using your calculator.

| $g(1.9)$ | $g(1.99)$ | $g(1.999)$ | $g(1.9999)$ | $g(2)$ | $g(2.0001)$ | $g(2.001)$ | $g(2.01)$ | $g(2.1)$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

**(2b)** What happens to $g(x)$ near $x = 2$? Does the function approach the same value as $x$ approaches 2 from above or below? Make a detailed sketch of the function $g(x)$ for $x \in [-5, 5]$.

**(3)** Consider $h(x) = \dfrac{\sin x}{x}$ as $x$ approaches 0 by sketching the graph of $h(x)$ in the vicinity of $x = 0$. Note that the function $h(x)$ is undefined for $x = 0$.

## 4.2 Formal definition

Suppose $f(x)$ is a real-valued function and $p$ is a real number. A function $f(x)$ is said to have a limit

$$\lim_{x\to p} = L,$$

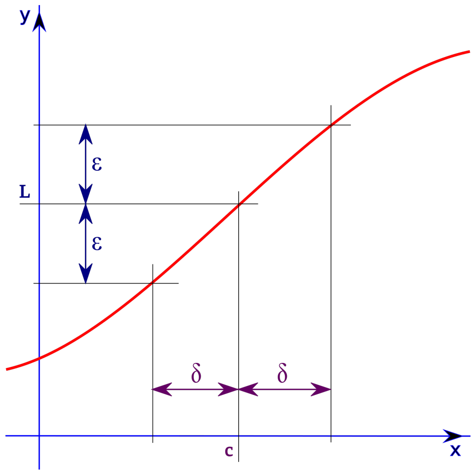
if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - p| < \delta$. This form of definition is sometimes called an epsilon-delta definition. Limits may be taken from below

$$\lim_{x\to p^-} f(x),$$

or from above

$$\lim_{x\to p^+} f(x).$$

If the two are equal, then "the" limit of the function $f(x)$ is said to exist at $x = p$.




### 4.2.1 Continuity

In the case where $f(c) = \lim_{x\to c} f(x)$, $f$ is said to be **continuous** at $x = c$.
In general, the limit of $f(x)$ as $x$ approaches $c$ is not necessarily equal to $f(c)$.

### 4.2.2 Useful identities

The following rules are valid if both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist

1. $\lim_{x\to c} [S \cdot f(x)] = S \cdot \lim_{x\to c} f(x)$ where $S$ is a scalar.
2. $\lim_{x\to c} b^{f(x)} = b^{\lim_{x\to c} f(x)}$ where $b$ is a positive real scalar.

3. $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

4. $\lim_{x \to c}(f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

5. $\lim_{x \to c}(f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

6. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

**Theorem:** If $f$ is a polynomial function and $a$ is a real number, then

$$\lim_{x \to a} f(x) = f(a).$$

**Sandwich theorem:** Suppose $f(x) \leq h(x) \leq g(x)$ for every $x$ in an open interval containing $a$. If $\lim_{x \to a} f(x) = L = \lim_{g \to a} g(x)$, then $\lim_{x \to a} h(x) = L$.

**Theorem:** (1) $\lim_{\theta \to 0} \sin\theta = 0$. (2) $\lim_{\theta \to 0} \cos\theta = 1$. (3) $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$. (4) $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta} = 0$. (In order to give a rigorous proof for (3) and (4), you would need to use the **Sandwich theorem**. )

## 4.3 Exercises

Find the limit, if it exists.

**1.** $\lim_{s \to 4} \dfrac{6s - 1}{2s - 9}$

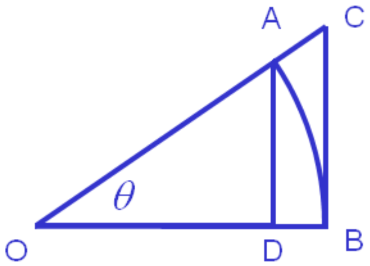
**2.** $\lim_{x \to 4^-} (x - \sqrt{16 - x^2})$

**3.** $\lim_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x - 2}$

**4.** Sketch the graph of the piecewise-defined function $f$ and, for the indicated value of $a$, find each limit, if it exists: (a) $\lim_{x \to a^-} f(x)$, (b) $\lim_{x \to a^+} f(x)$, (c) $\lim_{x \to a} f(x)$.

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 4 - x^2 & \text{if } x > 1 \end{cases} \quad \text{for } a = 1$$

**5.** Prove that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. [Hint: (1) Compare the lengths of $\overline{AD}$, $\overline{BC}$ and the circular arc $\widehat{AB}$ subtending the angle $\theta$. (2) Use the **Sandwich theorem**. ]




**6.** Prove that $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$. [Hint: $\sin^2\theta + \cos^2\theta = 1$]

# Chapter 5

# Differentiation

## 5.1 Examples

**(1a)** Consider $f(x) = \dfrac{x}{x^2+1}$. Compute $h(\Delta) = (f(x+\Delta) - f(x))/\Delta$ with $x = 2$ for various values of $\Delta$. Complete the following table using your calculator.

| $h(-0.5)$ | $h(-0.1)$ | $h(-0.01)$ | $h(0.01)$ | $h(0.1)$ | $h(0.5)$ |
|---|---|---|---|---|---|
| | | | | | |

**(1b)** Plot the function $f(x)$ for the range $x \in [1.5, 2.5]$.

**(1c)** Draw the straight lines connecting these pairs of points, $(x, f(x))$ and $(x+\Delta, f(x+\Delta))$ for these values of $\Delta$ in the above table. As you should see from your figure, $h(\Delta)$ gives the slope of the straight line connecting the two points, $(x, f(x))$ and $(x+\Delta, f(x+\Delta))$. What would be your guess for $\lim_{\Delta \to 0} h(\Delta)$? What is the meaning of this value?

**(2)** Carry out the same procedure as the previous problem for the function $g(x) = \dfrac{1}{|x|+1}$ at $x = 0$.

## 5.2 Definitions and properties

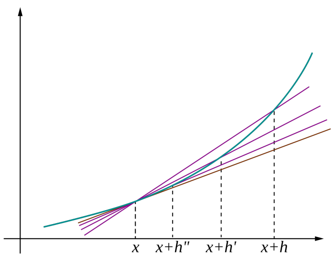
**Differentiation** is a method to compute the rate at which a dependent output, $f(x)$, changes with respect to the change in the independent input $x$. This rate of change is called the **derivative** of $f(x)$ with respect to $x$.

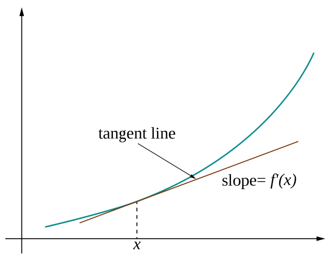
### 5.2.1 Derivative

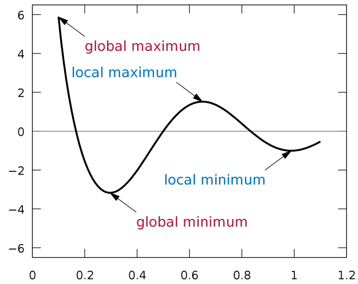
Formally, the derivative of a function represents an infinitesimal change in the function with respect to its variable. The derivative of a single-variable function $f(x)$ with respect the variable $x$ is defined as

$$f'(x) \equiv \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Note that $f'(x)$ is another function of $x$. If the limit exists, then $f(x)$ is differentiable at $x$.




$f'(a)$ gives the slope of the tangent line to the graph of $f(a)$ at the point of $x = a$. Therefore, if $f'(a) > 0$, the function is increasing in the vicinity of $x = a$. In contrast, if $f'(a) < 0$, the function is decreasing in the vicinity of $x = a$. If $f'(a) = 0$, the function has a local maximum or minimum at $x = a$. A point $x_0$ at which $f'(x_0) = 0$ is called a stationary point.

### 5.2.2 Linear approximation

The tangent line to $f(x)$ at $x = a$ can be written as

$$h(x) = f(a) + f'(a)(x - a).$$

$h(x)$ gives the best **linear approximation** to the function $f(x)$ in the region near $x = a$.

$$f(x) \approx f(a) + f'(a)(x - a).$$

### 5.2.3 Continuity and differentiability

Note that in order for the limit to exist, both $\lim_{\Delta h \to 0^+}$ and $\lim_{\Delta h \to 0^-}$ must exist and be equal, so the function must be continuous. If $y = f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. However, even if a function is continuous at a point, it may not be necessarily differentiable there. In summary: in order for a function $f(x)$ to have a derivative it is necessary for the function $f(x)$ to be continuous, but continuity alone is not sufficient.

### 5.2.4 The derivative as a function

Let $f(x)$ be a function that has a derivative at every point $a$ in the domain of $f(x)$. Recall that a function is a mapping. Because every point $a$ has a derivative, there is a function which sends the point $a$ to the derivative of $f(x)$ at $x = a$. This function is written as $f'(x)$ and is called the derivative function or the derivative of $f(x)$. The derivative of $f(x)$ collects all the derivatives of $f(x)$ at all the points in the domain of $f(x)$.

Sometimes $f(x)$ has a derivative at most, but not all, points of its domain. The function whose value at $x = a$ equals $f'(a)$ whenever $f'(a)$ is defined and is undefined elsewhere is also called the derivative of $f(x)$. It is still a function, but its domain is strictly smaller than the domain of $f(x)$.

Consider $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x \end{aligned}$$

Here is another example, consider $g(x) = \sin x$.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \sin x \left( \frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \frac{\sin h}{h} \\ &= \cos x \end{aligned}$$

### 5.2.5 Higher derivative

If the first derivative exists, the second derivative may be defined as

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

provided the second derivative is known to exist.

If $f''(a) > 0$, $f'(x)$ is an increasing function in the vicinity of $x = a$.

### 5.2.6 Second derivative test

If the function $f(x)$ is twice differentiable at a stationary point $x_0$, meaning that $f'(x_0) = 0$, then:

- If $f''(x_0) < 0$, then $f(x)$ has a local maximum at $x_0$.
- If $f''(x_0) > 0$, then $f(x)$ has a local minimum at $x_0$.
- If $f''(x_0) = 0$, then the second derivative test says nothing about the point $x_0$, which can possibly be an inflection point.

### 5.2.7 Derivatives of elementary functions

Note: you can show these derivatives of elementary functions using the definition of derivative.

- Derivatives of powers: If $f(x) = x^r$, where $r$ is any real number, then $f'(x) = rx^{r-1}$.
- Exponential and logarithmic functions:
  1. $\dfrac{d}{dx}e^x = e^x$
  2. $\dfrac{d}{dx}a^x = \ln(a)a^x$
  3. $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ for $x > 0$
  4. $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$
- Trigonometric functions:
  1. $\dfrac{d}{dx}\sin(x) = \cos(x)$
  2. $\dfrac{d}{dx}\cos(x) = -\sin(x)$
  3. $\dfrac{d}{dx}\tan(x) = \sec^2(x)$

### 5.2.8 Rules for finding the derivatives

Note: you can prove these rules using the definition of derivative.

- Constant rule: If $f(x)$ is constant, then $f'(x) = 0$.
- Sum rule: $(af + bg)' = af' + bg'$ for all functions $f$ and $g$ and real numbers $a$ and $b$.
- Product rule: $(fg)' = f'g + fg'$ for all functions $f$ and $g$.
- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ for all functions $f$ and $g$ where $g \neq 0$.
- Chain rule: If $f(x) = h(g(x))$, then $f'(x) = h'(g(x)) \cdot g'(x)$.

## 5.3 Physics intuition

In the case of a particle traveling in a straight line, its position, $x$, is given by $x(t)$ where $t$ is time and $x(t)$ means that $x$ is a function of $t$. The derivative of this function is equal to the infinitesimal change in quantity, $dx$, per infinitesimal change in time, $dt$ (of course, the derivative itself is dependent on time). This change in displacement per change in time is the velocity $v$ of the particle

$$\frac{dx}{dt} = v(t).$$

## 5.4 Exercises

Compute the first and second derivatives of the following functions.

**1.** $f(x) = 3x^2 - 4x + 1$.

**2.** $f(x) = \dfrac{x^2}{x-1} + x$.

**3.** $f(x) = 3\sin(x) + 2\cos(x)$.

**4.** Let $f(x) = \begin{cases} (2x-1)^3 & \text{if } x \geq 2, \\ 5x^2 + 34x - 61 & \text{if } x < 2. \end{cases}$

Determine if $f(x)$ is differentiable at $x = 2$.

**5.** $f(x) = x^3 - 4x$. (a) Find the equation of the tangent line to the graph of $f(x)$ at (2,0). (b) Find the points on the graph at which the tangent line is horizontal. Can you identify these points as local maximum or minimum? (c) Sketch the graph of $f(x)$ on the interval [-10,10].

# Chapter 6

# Integration

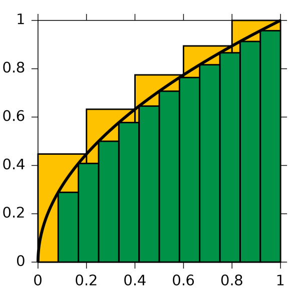
## 6.1 Example

In the example, we will try to find the area of the region in the $xy$-plane bounded by the graph of $f(x)$, the $x$ axis, and the vertical lines $x = a$ and $x = b$.

**(1a)** Consider the function $f(x) = \sqrt{x}$. Make a graph of $f(x)$ for $x \in [0,1]$.

**(1b)** Here is a simple way to get an approximation as shown in the figure. First, divide the range of $[0,1]$ into $N$ equally spaced intervals of width $\frac{1}{N}$. The $(N+1)$ end points of these $N$ intervals are $x_i = \frac{i}{N}$, for $i = 0, 2, \ldots, N$. One can obtain an approximation for the area by adding up all these rectangular areas. In this example, if you use the function value at the right end of each interval, the sum of all rectangular areas will be greater than the actual area. In contrast, if you use the function value at the left end of each interval, the sum of all rectangular areas will be smaller than the actual area. So, the true value of the area should be between these two sums. Use these two methods to obtain the sum of all rectangular areas for $N$=5,10,20,50,.... (It would be convenient to use EXCEL for this task!) You should observe that these two sums become closer to each other as $N$ increase.

**(1c)** What is your best approximation for the area?



**(2a)** Given $f(x) = x^2 + 3x - 1$, find a function $g(x)$ such that $g'(x) = f(x)$. $g(x)$ is called an **antiderivative** of $f(x)$.

**(2b)** Can you find other antiderivatives of $f(x)$?

**(2c)** Find the antiderivatives of the following functions: $\sin(x)$, $\exp(x)$, $\cos(x)$.

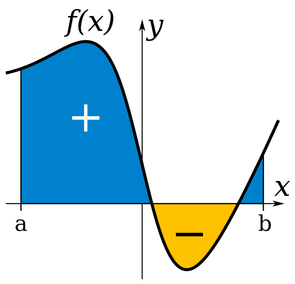
## 6.2 Definitions and properties

An integral is a mathematical object that can be interpreted as an area or a generalization of area. Integrals, together with derivatives, are the fundamental objects of calculus.

Given a function $f(x)$ of a real variable $x$ and an interval $[a, b]$ of the real line, the **definite integral**

$$\int_a^b f(x)dx$$

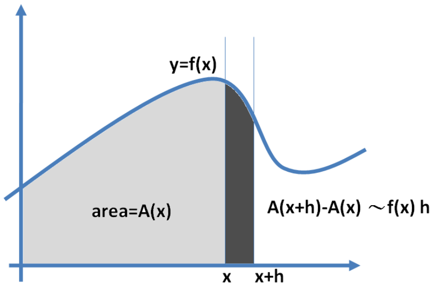
is defined informally to be the net signed area of the region in the $xy$-plane bounded by the graph of $f(x)$, the $x$ axis, and the vertical lines $x = a$ and $x = b$. Here, $a$ and $b$ are, respectively, referred as the upper and lower bounds of the integration. In the region where $f(x)$ is positive, the signed area defined as positive; in the region where $f(x)$ is negative, the signed area is negative.




### 6.2.1 Geometric intuition

Consider the graph of a continuous function $y = f(x)$ as shown in the figure. The signed area of the region beneath the curve of $f(x)$ between 0 and $x$ is denoted as a function $A(x)$, although we may not know a "formula" for the function $A(x)$.

Now suppose we wanted to compute the area under the curve between $x$ and $x + h$. We could compute this area by finding the area between 0 and $x + h$, then subtracting the area between 0 and $x$. In other words, this area would be $A(x + h) - A(x)$.




Similar to the example at the beginning of this chapter, we can use the rectangular area, given by the product of $h$ and $f(x)$, to approximate this area. Intuitively, the approximation will become very good as $h$ becomes smaller.

At this point we know that $A(x+h) - A(x)$ is approximately equal to $f(x) \cdot h$, and we intuitively understand that this approximation becomes better as $h$ grows smaller. In other words, $f(x) \cdot h \approx A(x+h) - A(x)$, with this approximation becoming an equality as $h$ approaches 0.

Divide both sides of this equation by $h$. Then we have

$$f(x) \approx \frac{A(x+h) - A(x)}{h}.$$

As $h$ approaches 0, the right hand side of this equation is simply the derivative $A(x)$ of the area function $A(x)$. The left-hand side of the equation simply remains $f(x)$, since no $h$ is present.

We have shown informally that $f(x) = A'(x)$. In other words, the derivative of the area function $A(x)$ is the original function $f(x)$. Or, to put it another way, the area function is simply the **antiderivative** of the original function.

What we have shown is that, intuitively, computing the derivative of a function and finding the area under its curve are "opposite" operations. This is the crux of the Fundamental Theorem of Calculus.

### 6.2.2 First Fundamental Theorem of Calculus

Let $f$ be a continuous real-valued function defined on a closed interval $[a, b]$. Let $F$ be the function defined, for all $x$ in $[a, b]$, by

$$F(x) = \int_a^x f(t)\, dt.$$

Then, $F$ is continuous on $[a, b]$, differentiable on the open interval $(a, b)$, and

$$F'(x) = f(x)$$

for all $x$ in $(a, b)$.

### 6.2.3 Second Fundamental Theorem of Calculus

Let $f$ be a real-valued function defined on a closed interval $[a, b]$ that admits an **antiderivative** $g$ on $[a, b]$. That is, $f$ and $g$ are functions such that for all $x$ in $[a, b]$,

$$f(x) = g'(x).$$

If $f$ is integrable on $[a, b]$ then

$$\int_a^b f(x)\, dx = g(b) - g(a).$$

Note that when an antiderivative $g$ exists, then there are infinitely many antiderivatives for $f$, obtained by adding to $g$ an arbitrary constant. Also, by the first part of the theorem, antiderivatives of $f$ always exist when $f$ is continuous.

**An important note: You should consult with your calculus textbook for rigorous proof for these theorems!!**

## 6.3 Physics intuition

The velocity of a particle is defined as the derivative of the displacement with respect to time.

$$v(x) = \frac{dx}{dt}$$

Rearranging this equation, it follows that:

$$dx = v(t)dt.$$

By the logic above, a change in $x$ (or $\Delta x$) is the sum of the infinitesimal changes $dx$. It is also equal to the sum of the infinitesimal products of the derivative and time. This infinite summation is integration; hence, the integration operation allows the recovery of the original function from its derivative. As one can reasonably infer, this operation works in reverse as we can differentiate the result of our integral to recover the original.

### 6.3.1 Antiderivatives

In contrast to a definite integral specified by given upper and lower bounds, **indefinite integral** is the process of finding the set of all antiderivative of $f(x)$.

$$\int f(x)\, dx = F(x) + C,$$

where

$$F'(x) = \frac{d}{dx}F(x) = f(x).$$

The expression $F(x) + C$ is the general antiderivative of $f$.

Here are some useful theorems that you can easily prove.

- $\int dx = x + c$
- If $n$ is a real number,

$$\int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1, \\ \ln|x| + C \text{ if } n = -1. \end{cases}$$

- The general antiderivative of a function multiplied by a constant is the constant multiplied by the general antiderivative of the function.

$$\int af(x)dx = a \int f(x)\, dx$$

- If $f$ and $g$ are defined on the same interval, then:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\, dx$$

- If $f_1, f_2, \cdots, f_n$ are defined on the same interval,

$$\begin{aligned} &\int \left[c_1 f_1(x) + c_2 f_x(x) + \cdots + c_n f_n(x)\right] \, dx \\ &= c_1 \int f_1(x)\, dx + c_2 \int f_2(x)\, dx + \cdots + c_n \int f_n(x)\, dx, \end{aligned}$$

  where $c_1, c_2, \cdots, c_n$ are constants.

Some other useful formulas, where $a$ is a constant:

- $\int \sin(au)\, du = -\frac{1}{a} \cos(au) + C$
- $\int \cos(au)\, du = \frac{1}{a} \sin(au) + C$
- $\int \exp(au)\, du = \frac{1}{a} \exp(au) + C$

## 6.4 Exercises

Evaluate.

**1.** $\int (3x^4 + 2x^3 - x)\, dx$

**2.** $\int_0^1 (2x - 1)(5x + 1)\, dx$

**3.** $\int_0^\pi \sin nx\, dx$, where $n$ is a constant.

**4.** $\int_0^\pi \cos x\, dx$

**5.** $\int_{-100}^{100} (3x^5 + 2x^3 - x)\, dx$

# Chapter 7

# Useful tools

## 7.1 GNUPLOT

Gnuplot is a powerful freely distributed software for data and function plotting. Here is the homepage of GNUPLOT, [ **http://www.gnuplot.info/** ]. It was originally intended as to allow scientists and students to visualize mathematical functions and data. GNUPLOT supports many types of plots in either 2D or 3D. It can draw using lines, points, boxes, contours, vector fields, surfaces, and various associated text. It also supports various specialized plot types. GNUPLOT supports many different types of output, including interactive screen terminals and output to many file formats (eps, fig, jpeg, LaTeX, metafont, pbm, pdf, png, postscript, svg, ...). The software has been supported and under development since 1986.

First, we suggest you to go to the **demos** section and see what the software can do for you. In addition, you can find a brief manual and tutorial at this web site, [ **http://www.duke.edu/~hpgavin/gnuplot.html** ]. Written in a very simple way that is very easy to follow, it begins with how to install and to start GNUPLOT on your computer.

You should install the software at your own computer. It is very simple. Go to the **download** section and get the current gnuplot version from the primary download site on SourceForge. Download and unzip the binary package, then you can use one of the executable files in the binary directory. In addition, familiarize yourself with the software. At least, you should learn how to make 2D and 3D plots for some elementary functions and polynomial functions. This is a very useful tool for your future study in mathematics and physics. We encourage you to explore more the capabilities of GNUPLOT.

### 7.1.1 Exercises

**1.** Generate a plot of the function $\sin(x)$.

**2.** Generate a plot of a polynomial of your choice.

**3.** Generate a plot of the function $f(x) = \sin(\exp(x))$ for $x \in [0, 5]$. Make sure that you use enough points to show all oscillations of the function in the range.

**4.** Can you output the plot to a file?

## 7.2 Wolfram|Alpha

Wolfram|Alpha is a web resource built by Wolfram Research Company, the producer of the powerful mathematical software, **Mathematica**. Here is the homepage of Wolfram|Alpha [ **http://www.wolframalpha.com/**].

Wolfram|Alpha aims to *"bring expert-level knowledge and capabilities to the broadest possible range of people - spanning all professions and education levels"*. It integrates several online resources also previously produced by Wolfram Research, including the Wolfram Demonstrations Project [ **http://demonstrations.wolfram.com/** ] and the Online Integrator [ **http://integrals.wolfram.com/** ]. Unlike the two precursors requiring a special syntax as used in **Mathematica**, Wolfram|Alpha accepts free-form inputs. It serves as a knowledge engine, similar to how Wikipedia works, but more interactive in providing visual demonstrations. We suggest you to explore the **examples by topic** section and learn how to use this powerful tool.

### 7.2.1 Exercises

**1.** Input `exp` and see what you get. You will get information about exponential function $e^x$. Try to change the range of the plots for the function.

**2.** Input `projectile motion` and see what you get. Change the initial speed and the release angle for the trajectory.

**3.** Input `integrate sin(x)`.

**4.** Input `integrate sin(x**2)` and see what you get.

**5.** Try to find the root of $x - \cos(x) = 0$.

# References


1. H. Benson, *University Physics* (Revised Edition), Wiley, 1995.

2. Weisstein, Eric W. "Topics in a Pre-Calculus Course." From *MathWorld* – A Wolfram Web Resource.
   `http://mathworld.wolfram.com/classroom/classes/Pre-Calculus.html`

3. WIKIPEDIA *The Free Encyclopedia*.
   `http://en.wikipedia.org/wiki`

4. Gnuplot homepage:
   `http://www.gnuplot.info`

5. Wolfram|Alpha homepage:
   `http://www.wolframalpha.com/`

6. Wolfram Demonstrations Project
   `http://demonstrations.wolfram.com/`

7. Wolfram Mathematica Online Integrator
   `http://integrals.wolfram.com/`